Prismatic Approaches to Regularity and Perfectoid Towers

プリズムを用いた正則性とパーフェクトイド塔の研究

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Introduction: Mixed Characteristic

Background

- **Characteristic 0 (e.g., complex varieties):** Analytic methods (e.g., resolution of singularities, vanishing theorems).
- **Positive characteristic ($p > 0$) (e.g., varieties over finite fields):** The Frobenius map $F(x) = x^p$ plays a central role (e.g., Kunz's theorem, F-singularities).
- **Mixed characteristic (e.g., p-adic formal schemes):** There are neither analytic methods nor a Frobenius map, but it connects both worlds.

Mixed Characteristic

$$\mathscr{X} = \mathrm{Spf}(R)$$

Generic Fiber ↙ ↘ Special Fiber

Characteristic 0

$$\mathscr{X}_\eta = \mathrm{Spec}(R[1/p])$$

Positive Characteristic

$$\mathscr{X}_{p=0} = \mathrm{Spec}(R/pR)$$

Introduction: Recent developments in perfectoid methods

- **Perfectoid spaces (Scholze, 2012):** Introduced perfectoid rings, which generalize perfect rings in characteristic p, and relate mixed characteristic to characteristic p via tilting.
- **Direct Summand Conjecture (André, 2018):** Proved using perfectoid techniques; this got modern mixed characteristic commutative algebra started.
- **Prisms (Bhatt–Scholze, 2022)[1]:** Generalize perfectoid rings and introduce prismatic cohomology, a new integral p-adic cohomology theory.
- **Perfectoid singularities (BIM, BMPSTWW):** Introduced mixed characteristic analogues of "F-singularities" using perfectoid techniques.
- **Perfectoid towers (Ishiro–Nakazato–Shimomoto, 2025)[2]:** Approximate perfectoid rings by sequences of (Noetherian) rings.

[1] Annals of Mathematics, 196(3) (2022) 1135–1275.

[2] Algebra & Number Theory, 19(12) (2025) 2307–2358

Introduction: Summary of Results

Main Theme of this Thesis

Study commutative algebra in mixed characteristic using prismatic methods.

This thesis has two main parts:

1. **Prismatic Kunz's Theorem**
   - Characterizes regularity of Noetherian local rings via the faithful flatness of the Frobenius lift φ on a prismatic complex $\Delta_{R/A}$.
2. **Perfectoid towers from prisms**
   - Constructs perfectoid towers from a large class of prisms.
   - Provides systematic examples of perfectoid towers and their tilts beyond the (log-)regular setting.

Prismatic Kunz's Theorem: Kunz and p-adic Kunz

Kunz's Theorem (Kunz, 1969)[3]

Let R be a Noetherian ring of char $p > 0$. TFAE:

1. R is regular.
2. $F\colon R \to R$ is (faithfully) flat.

p-adic Kunz's Theorem (Bhatt–Iyengar–Ma, 2019)[4]

Let R be a Noetherian ring with $p \in \mathrm{Jac}(R)$. TFAE:

1. R is regular.
2. There exists a perfectoid ring S and a faithfully flat map $R \to S$.

In mixed characteristic, singularities can be detected via morphisms to perfectoid rings rather than via Frobenius.

[3] American Journal of Mathematics, 91 (1969) 772–784.

[4] Communications in Algebra, 47(6) (2019) 2367–2383.

Prismatic Kunz's Theorem: Prismatic Approach

- p-adic Kunz's theorem uses perfectoid rings but does not explicitly involve an endomorphism like Frobenius.
- **Heuristic (following Bhatt):** A "mixed characteristic Frobenius" for a ring R is the Frobenius lift φ on a prismatic cohomology $\Delta_{R/A}$.

Definition (Prisms: p-torsion-free case)

A pair (A, I) is a *(bounded) prism* if:

1. A has a Frobenius lift $\varphi_A \colon A \to A$, i.e., a ring endomorphism such that $\varphi_A(a) \equiv a^p \mod pA$ for all $a \in A$.
2. $I \subset A$ is an invertible ideal of A.
3. A is (p, I)-adically complete and A/I has bounded p^∞-torsion.
4. $p \in I + \varphi_A(I)A$.

- For example, $(A, I) = (\mathbb{Z}_p[|T_1, \ldots, T_n|], (p - f))$ for some $f \in (T_1, \ldots, T_n)$ and $\varphi_A(T_i) := T_i^p$.

Prismatic Kunz's Theorem: Prismatic Cohomology

Prismatic Cohomology (Bhatt–Scholze)

Let (A, I) be a bounded prism (e.g., $(\mathbb{Z}_p[|T_1, \ldots, T_n|], (p - f))$). For an A/I-algebra $A/I \to R$, we define the *prismatic cohomology* $\Delta_{R/A}$ of $A/I \to R$. Naively, this is a commutative algebra object in the derived category $D(A)$ of A-modules:

$$\Delta_{R/A} \in D(A).$$

Moreover, $\Delta_{R/A}$ is equipped with an endomorphism $\varphi \colon \Delta_{R/A} \to \Delta_{R/A}$, called the *Frobenius lift*.

- Prismatic cohomology is defined as the non-additive derived functor of the cohomology of the prismatic site $(R/A)_{\Delta}$ for smooth A/I-algebras R.

Prismatic Kunz's Theorem: Main Theorem 1

Question: Can we characterize the regularity of R in terms of the Frobenius lift φ on $\Delta_{R/A}$ for some A?

Theorem (Prismatic Kunz's Theorem (LCI case), I.–Nakazato[5])

Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring of residue characteristic p. Suppose that R is a complete intersection. Then there exist a bounded prism (A, I) and a surjection $A/I \twoheadrightarrow R = A/(I, f_1, \ldots, f_r)$ such that TFAE:

1. *R is regular.*
2. *The Frobenius lift $\varphi\colon \Delta_{R/A} \to \Delta_{R/A}$ is (p-completely) faithfully flat.*

- The proof needs "regular prisms" and the faithful flatness of prismatic complexes.
- This extends to general complete Noetherian local rings of residue characteristic p by using the derived quotient $A/^L(I, f_1, \ldots, f_r)$.

[5] Journal of Algebra, 693 (2026) 732–759.

Prismatic Kunz's Theorem: Key Ingredients

Lemma

Let $(R, \mathfrak{m}, k)$ *be a complete Noetherian local ring of residue characteristic* p*. By Cohen structure theorem, we take a surjection*

$$A := C(k)[|T_1, \dots, T_e|] \twoheadrightarrow R,$$

where $e := \dim_k \mathfrak{m}/\mathfrak{m}^2$ *and* $C(k)$ *is the Cohen ring of* k*. Then there exists* $f \in (T_1, \dots, T_e)$ *such that* $(A, (p - f))$ *is a bounded prism and* $p - f \mapsto 0$*.*

Lemma

Let (A, I) *be a bounded prism and let* $f_1, \dots, f_r$ *be a regular sequence in* A/I*. Set* $R := A/(I, f_1, \dots, f_r)$*. Then* $\Delta_{R/A}$ *is concentrated in degree* 0*, and the structure morphism*

$$R \to \overline{\Delta}_{R/A} := \Delta_{R/A} \otimes_A A/I$$

is p-completely faithfully flat.

Prismatic Kunz's Theorem: Proof Sketch

Regularity $\Rightarrow$ faithful flatness

If R is regular, then $A/I \cong R$. Thus, $\Delta_{R/A} \cong A$ and the Frobenius lift φ_A is faithfully flat.

Faithful flatness $\Rightarrow$ regularity

If φ is faithfully flat, then so is the canonical morphism

$$\Delta_{R/A} \to \operatorname*{colim}_{e \geq 0} \varphi_*^e \Delta_{R/A}.$$

Reducing modulo I and using the lemma, we obtain a faithfully flat morphism

$$R \to \left(\operatorname*{colim}_{e \geq 0} \varphi_*^e \Delta_{R/A} \right) / I.$$

The target is a perfectoid ring, so R is regular by p-adic Kunz's theorem.

$\square$

Perfectoid Towers: Motivation

Observation

- Perfectoid rings are powerful but non-Noetherian.
- Many perfectoid rings over Noetherian bases arise as colimits of sequences of Noetherian rings, e.g., $\mathbb{Z}[p^{1/p^\infty}]^{\wedge_p}$.
- We want to use tilting techniques for Noetherian rings directly.

Perfectoid tower (Ishiro–Nakazato–Shimomoto)

A *perfectoid tower* is a sequence of p-adically complete rings $R_0 \to R_1 \to R_2 \to \cdots$ satisfying certain axioms. For example, R_1 has an ideal I_1 satisfying $I_1^p = pR_1$ and, for each $i \geq 0$, a commutative diagram:

$$\begin{array}{ccc} R_{i+1}/I_1 R_{i+1} & \xrightarrow{a \mapsto a^p} & R_{i+1}/pR_{i+1} \\ & \searrow^{\exists \cong} & \uparrow \\ & & R_i/pR_i. \end{array}$$

Perfectoid Towers: Consequences of Perfectoid Towers

Example

- $\mathbb{Z}_p \to \mathbb{Z}_p[p^{1/p}] \to \mathbb{Z}_p[p^{1/p^2}] \to \cdots$.
- Any complete (log-)regular local ring R of mixed characteristic, e.g.,

$$R \cong W[|\underline{T}|]/(p-f) \to W[|p^{1/p}, \underline{T}^{1/p}|]/(p-f) \to \cdots .$$

- The colimit $R_\infty := (\mathrm{colim}_i\, R_i)^{\wedge_p}$ is a perfectoid ring.
- We can take the *tilt* of the tower:

$$R_0^{s.\flat} \to R_1^{s.\flat} \to R_2^{s.\flat} \to \cdots ,$$

which is a tower of $\mathbb{F}_p$-algebras with an element $p^{s.\flat} \in R_0^{s.\flat}$.
- An isomorphism $R_i^{s.\flat}/p^{s.\flat} \cong R_i/p$ holds for each i.
- $\{R_i\}_{i\geq 0}$ is a *lim Cohen–Macaulay sequence* if R_0 is a complete Noetherian local domain of mixed characteristic with perfect residue field (Bhatt–Hochster–Ma, Ishiro–Shimomoto).

Perfectoid Towers: Main Theorem 2

Issue: Previous constructions were restricted to (log-)regular local rings, and the proofs were often ad hoc.
We construct perfectoid towers from *prisms* systematically.

Theorem (Perfectoid Towers from Prisms, I.[6])

Let (A, dA) be a prism such that p, d is a regular sequence on A and A/pA is p-root closed in $(A/pA)[1/d]$. Then the sequence

$$A/dA \xrightarrow{\varphi} A/\varphi(d)A \xrightarrow{\varphi} A/\varphi^2(d)A \to \cdots$$

forms a perfectoid tower over A/dA. Its tilt is given by

$$(A/pA)^{\wedge_d} \xrightarrow{F} (A/pA)^{\wedge_d} \xrightarrow{F} (A/pA)^{\wedge_d} \to \cdots,$$

where $(A/pA)^{\wedge_d}$ is the d-adic completion of A/pA and F is the Frobenius map.

[6] to appear in Nagoya Mathematical Journal.

Perfectoid Towers: Concrete Examples

As a special case, we consider prisms of the form $(A, I) = (R_0[|T|], (p - T))$, where R_0 is a ring equipped with a Frobenius lift φ satisfying mild conditions.

Corollary

Let $(R_0, \mathfrak{m})$ be a p-torsion-free complete Noetherian local ring of residue characteristic p such that R_0/pR_0 is reduced. If it admits a Frobenius lift φ, then there exists a perfectoid tower over R_0 of the form

$$R_0 \xrightarrow{\varphi \otimes \mathrm{inclusion}} R_0 \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p}] \xrightarrow{\varphi \otimes \mathrm{inclusion}} R_0 \otimes_{\mathbb{Z}} \mathbb{Z}[p^{1/p^2}] \to \cdots$$

whose tilt is given by

$$R_0/pR_0[|T|] \xrightarrow{F} R_0/pR_0[|T|] \xrightarrow{F} R_0/pR_0[|T|] \to \cdots .$$

Our method recovers the (log-)regular computations of perfectoid towers due to Ishiro–Nakazato–Shimomoto.

Perfectoid Towers: Examples

Explicitly, our method produces perfectoid towers over various singular rings.

Example

- $\mathbb{Z}_p[|X, Y, Z, W|]/(XY, p - ZW)$: a ramified lci non-domain.
- $\mathbb{Z}_p[|S, ST, ST^3, ST^4|]$: a non-CM, non-normal domain.
- $\mathbb{Z}_p[|S^2, S^3|]$: an lci domain that is not normal.
- $\mathbb{Z}_p[|X, Y, Z|]/(XY, YZ)$: a reduced, non-CM, and non-domain.
- $\mathbb{Z}_2[|X, Y, Z|]/(X^3 + Y^4 + Z^5, Y^8 + X^3Y^4 + X^6)$: an unramified lci domain that is not log-regular.
- $R(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a canonical lift of an ordinary abelian variety A over a perfect field of characteristic p and $\mathcal{L}$ is an ample line bundle on $\mathcal{A}$ (the section ring of $\mathcal{L}$): a non-CM normal domain if $\dim A > 1$.

Summary and Future Outlook

Summary

1. **Prismatic Kunz's Theorem:** Characterized regularity using the Frobenius lift on prismatic cohomology.
2. **Perfectoid towers:** Developed a general construction from prisms, applicable even to singular rings.

Future Directions

- Study mixed characteristic singularities defined via the Frobenius lift on prismatic cohomology.
- Apply perfectoid towers to mixed characteristic singularities.
- More broadly, refine and extend mixed characteristic singularity theory using prismatic and perfectoid methods.