On the derived deformation functor of Frobenius liftings

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Frobenius Liftings

(k, F): perfect field of characteristic $p > 0$ with Frobenius F.
(W_n, F_n): n-truncated Witt vectors over k with Frobenius lift F_n.
(X, F): scheme over k with Frobenius F.

Definition

A **Frobenius lifting** of (X, F) over W_n is a pair (X_n, F_n) where X_n is a flat W_n-scheme and $F_n : X_n \to X_n$ is a lift of the Frobenius morphism compatible with F_n on W_n such that the base change to k recovers (X, F).

- If X is smooth projective variety, the existence of Frobenius liftings over W_2 implies Bott vanishing:

$$H^j(X, \Omega^i_{X/k} \otimes_{\mathcal{O}_X} L) = 0 \quad \text{for } j > 0 \text{ and } L \text{ ample.}$$

Classical Obstruction Theory

Theorem (Nori–Srinivas)[1]

X: **smooth** variety over k.
(X_n, F_n): Frobenius lifting of X over W_n.
The obstruction to Frobenius liftability of (X_n, F_n) to W_{n+1} lies in

$$\mathrm{ob}(X_n, F_n) \in H^1(X, T_X \otimes_{\mathcal{O}_X} B^1_X) \cong \mathrm{Ext}^1_X(\Omega^1_{X/k}, B^1_X).$$

If it vanishes, the isomorphism classes of such Frobenius liftings is a torsor under $H^0(X, T_X \otimes_{\mathcal{O}_X} B^1_X)$.

- $T_X := \mathcal{H}\mathrm{om}_X(\Omega^1_{X/k}, \mathcal{O}_X)$ and $B^1_X := \mathrm{coker}(\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X)$.
- This proof relies on the Frobenius liftability of **smooth affine** varieties and does not generalize to singular schemes.

[1] Compositio Mathematica, 64.2 (1987), 191–212.

Derived Deformation Functors

Definition (Grothendieck[2]–Schlessinger[3]–Lurie[4])

$\mathrm{CAlg}_{W//k}^{an,\mathrm{art}}$: the ∞-category of animated Artinian local $W(k)$-algebras.
A **formal moduli problem** is a functor

$$F\colon \mathrm{CAlg}_{W//k}^{an,\mathrm{art}} \to \mathrm{Ani}$$

satisfying certain properties (see the next slide).

- Brantner–Taelman[5]: The functor Def_X which classifies liftings of a (derived) k-scheme X is a formal moduli problem.

[2] Séminaire Bourbaki, Vol.5. Soc. Math. France, Paris, 1960, Exp. No. 195, 369–390
[3] Transactions of the American Mathematical Society, 130.2 (1968), 208–222
[4] Spectral Algebraic Geometry
[5] arXiv:2407.09256

Required Properties of Formal Moduli Problems

A formal moduli problem F is defined by the following properties:

1. The anima $F(k)$ is contractible.
2. For $B \to A \leftarrow C$ in $\mathrm{CAlg}^{an,\mathrm{art}}_{W//k}$, the canonical map

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

   in Ani is an equivalence if the maps $\pi_0(B) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(A)$ are surjective.

We can take the following pullback diagram in $\mathrm{CAlg}^{an,\mathrm{art}}_{W//k}$:

$$\begin{tikzcd} W_{n+1} \arrow[r, "\text{mod } p^n"] \arrow[d] & W_n \arrow[d] \\ k \arrow[r] & k \oplus (k[1]) \end{tikzcd}$$

where $k \oplus (k[1])$ is the trivial square zero extension of k by $k[1] \in \mathcal{D}(k)$.

Formal Moduli Problem of Schemes

Apply the formal moduli problem Def_X to the pullback square above and compute the value $\mathrm{Def}_X(k \oplus (k[1]))$:

Theorem (Brantner–Taelman)

- $\mathrm{Def}_X(k \oplus (k[1])) \simeq \mathrm{Map}_X(L_{X/k}, \mathcal{O}_X[2])$.
- X_n: a lifting over W_n of a (derived) k-scheme X. Then $\mathrm{Def}_X(W_{n+1}) \simeq \mathrm{Def}_X(W_n) \times_{\mathrm{Def}_X(k\oplus(k[1]))} \{*\}$. In particular, we have a fiber sequence of sets

$$\pi_0(\mathrm{Def}_X(W_{n+1})) \to \pi_0(\mathrm{Def}_X(W_n)) \to \mathrm{Ext}^2_X(L_{X/k}, \mathcal{O}_X)$$

where the fiber is taken over the zero element in $\mathrm{Ext}^2_X(L_{X/k}, \mathcal{O}_X)$; the image of X_n in the last term is an obstruction class.

This provides a conceptual framework for studying liftings of schemes. We want to construct a similar one for Frobenius liftings.

Categorical Setting for Frobenius Liftings

Definition

The ∞-category $\mathrm{End}(\mathrm{CAlg}^{an})_{(W,F)//(k,F)}$ is defined as follows:

- Objects: endomorphisms $\varphi\colon A \to A$ of animated rings A equipped with a morphism $W \to A \to k$ such that φ is compatible with the Frobenius lift $F\colon W \to W$ and the Frobenius map $F\colon k \to k$.
- Morphisms: natural transformations between such endomorphisms.

We can define the Artinian objects in this ∞-category and denote the full ∞-subcategory spanned by them as $\mathrm{End}(\mathrm{CAlg}^{an})^{\mathrm{art}}_{(W,F)//(k,F)}$.

- For example, the Frobenius lift $F_n\colon W_n \to W_n$ on the n-truncated Witt ring W_n belongs to this ∞-category.
- All limits and colimits in this ∞-category are computed termwise.

Main Theorem: Formal Moduli Problem

Set $B^1_X := \mathrm{cofib}(\mathcal{O}_X \xrightarrow{F} F_*\mathcal{O}_X)$ in $\mathcal{D}(X)$.

Theorem (I.)

There exists a formal moduli problem of Frobenius liftings of X:

$$\mathrm{Def}_{(X,F)} \colon \mathrm{End}(\mathrm{CAlg}^{an})^{\mathrm{art}}_{(W,F)//(k,F)} \to \mathrm{Ani}\,.$$

- $\mathrm{Def}_{(X,F)}(k \oplus k[1], F \oplus (F[1])) \simeq \mathrm{Map}_X(L_{X/k}, B^1_X)$.
- For $\mathrm{mod}\, p^n$-map, we have a fiber sequence of sets

$$\begin{aligned}\pi_0(\mathrm{Def}_{(X,F)}(W_{n+1}, F_{n+1})) &\to \pi_0(\mathrm{Def}_{(X,F)}(W_n, F_n)) \\ &\to \mathrm{Ext}^1_X(L_{X/k}, B^1_X)\end{aligned}$$

where the fiber is taken over the zero element in $\mathrm{Ext}^1_X(L_{X/k}, B^1_X)$.

Main Theorem: Obstruction Theory

Corollary (I.)

(X_n, F_n): Frobenius lifting of X over W_n.
The obstruction to Frobenius liftability of (X_n, F_n) to W_{n+1} lies in

$$\mathrm{ob}(X_n, F_n) \in \mathrm{Ext}^1_X(L_{X/k}, B^1_X).$$

If it vanishes, the isomorphism classes of such Frobenius liftings is a torsor under the hom-set $\mathrm{Hom}_X(L_{X/k}, B^1_X)$.

Remark

- This theory works for any derived scheme X over a (not necessarily perfect) field k of characteristic $p > 0$.
- This is self-contained and does not depend on the classical obstruction theory such as Nori–Srinivas and Illusie (in spite of using cotangent complexes).

Corollary and Future Works

Examples

- Smooth affine k-schemes
- Perfect k-schemes
- F-split smooth k-schemes with trivial cotangent bundle (Nori–Srinivas)

admit Frobenius liftings over W_n for all n.
Any F-split derived scheme has a lifting over W_2.

- Reprove other previous results on Frobenius liftings using this framework more conceptually.
- Study Frobenius liftability of singular and derived schemes.
- Compatibility of the Nori–Srinivas obstruction theory in the smooth case. Is the obstruction $\mathrm{ob}(X, F) \in \mathrm{Ext}_X(\Omega^1_{X/k}, B^1_X)$ of Frobenius liftability over W_2 the same as the Cartier exact sequence $0 \to B^1_X \to Z^1_X \xrightarrow{C} \Omega^1_{X/k} \to 0$?