「REGULAR RINGS AND PERFECT(OID) ALGEBRAS」の紹介

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This is a note for the 18-th summer school on Commutative algebra at the Tokyo Institute of Technology to introduce the paper [BIM].

Convention

- All rings are assumed to be commutative and contains unity.
- Let p denote a fixed prime number.

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Introduction

The paper [BIM] explores some homological properties of perfect(oid) algebras over commutative noetherian rings. One of results is Kunz's theorem in mixed characteristic case.

Recall that Kunz's theorem asserts that a noetherian $\mathbb{F}_p$-algebra R is regular if and only if the Frobenius map $R \to R$ is flat. One can reformulate this result as the following assertion: such an R is regular exactly when there exists a faithfully flat map $R \to A$ with A perfect (Proposition 1.6). Our p-adic generalization is the following:

Theorem A (see Theorem 1.10)**.** *Let R be a noetherian ring such that p lies in the Jacobson radical of R (for example, R could be p-adically complete). Then R is regular if and only if there exists a faithfully flat map $R \to A$ with A perfectoid.*

§1 is devoted to Kunz's theorem in both positive characteristic case and mixed characteristic case.

In §2, we introduce the results about finiteness of flat dimension. In fact, the original proof of Theorem A in [BIM] is based on results in this section.

In §3, we focus on two algebras: the absolute integral closure R^+ and the perfect closure R_{perf}. In addition, we will introduce the notions of *proregular sequences* and *weakly proregular sequences*. It will turn out that systems of parameters of R are weakly proregular on R^+ and on R_{perf} under some conditions (Proposition 3.9). Because of this, we obtain a criterion of regulaness using the vanishing of $\text{Tor}_i^R(R^+, k)$ or $\text{Tor}_i^R(R_{\text{perf}}, k)$ (Theorem 3.11)

1. Kunz's theorem

In this section we give a quick but reasonably detailed overview of the proof of Kunz's theorem in mixed characteristic case.

First in §1.1 we recall classical Kunz's theorem and some applications. This subsection ends with reformulation of Kunz's theorem

After understanding the proof of reformulation of Kunz's theorem in positive characteristic, we proceed in §1.2 to generalizing to mixed characteristic, using the notion of perfectoid rings.

1.1. Positive characteristic.

Throughout this subsection, let R denote a noetherian $\mathbb{F}_p$-algebra.

Let us first briefly recall Kunz's theorem.

Theorem 1.1 ([Kunz69, Theorem 2.1, Corollary 2.7])**.** *The following conditions are equivalent.*

(1) *R is regular.*
(2) *The absolute Frobenius $\varphi\colon R \to R$ is flat.*

Sketch. We may assume that R is complete local (here we use the fact that a local homomorphism $A \to B$ of noetherian local rings is (faithfully) flat if and only if so is $\widehat{A} \to \widehat{B}$, which is a consequence of local criterion of flatness). Let $k := R/\mathfrak{m}_R$.

(1) $\Rightarrow$ (2): By Cohen's structure theorem, we may assume that $R = k[[X_1, \ldots, X_n]]$, where $n = \dim R$. Then the canonical injection $R^p \hookrightarrow R$ can be decomposed as

$$R^p = k^p[[X_1^p, \ldots, X_n^p]] \hookrightarrow k[[X_1^p, \ldots, X_n^p]] \hookrightarrow k[[X_1, \ldots, X_n]] = R.$$

Since $k[[X_1, \ldots, X_n]]$ is free over $k[[X_1^p, \ldots, X_n^p]]$ on the basis $\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid 0 \leq \alpha_i \leq p-1\}$ and since the flatness of $k^p \hookrightarrow k$ implies that of $k^p[[X_1^p, \ldots, X_n^p]] \hookrightarrow k[[X_1^p, \ldots, X_n^p]]$, it follows that $R^p \hookrightarrow R$ is flat.

(2) $\Rightarrow$ (1): Let $x_1, \ldots, x_r$ be a minimal basis of $\mathfrak{m}_R$. Then by Cohen's structure theorem, we have a surjection

$$S := k[[X_1, \ldots, X_r]] \twoheadrightarrow R, \quad X_i \mapsto x_i.$$

Let $\mathfrak{a}$ be the kernel. Then, for each p-power $q = p^\nu$, the surjection induces the short exact sequence

$$0 \longrightarrow (\mathfrak{a} + \mathfrak{m}_S^q)/\mathfrak{m}_S^q \longrightarrow S/\mathfrak{m}_S^q \longrightarrow R/\mathfrak{m}_R^q \longrightarrow 0.$$

Using the notion of independendence in the sense of Lech, we can prove that $l_S(S/\mathfrak{m}_S^q) = q^r = l_R(R/\mathfrak{m}_R^q) = l_S(R/\mathfrak{m}_R^q)$, so that $(\mathfrak{a} + \mathfrak{m}_S^q)/\mathfrak{m}_S^q = 0$, i.e., $\mathfrak{a} \subset \mathfrak{m}_S^q$. Since this holds true for any p-power $q = p^\nu$, one has

$$\mathfrak{a} \subseteq \bigcap_{\nu > 0} \mathfrak{m}_S^q = (0).$$

Thus $R \cong S = k[[X_1, \ldots, X_r]]$, which is a regular local ring. □

Let us mention applications of Kunz's theorem.

The fact that a localization of a regular local ring is again regular is proved by (Auslander-Buchbaum-)Serre's theorem for regular local rings. But in positive characteristic case, this fact is an immediate consequence of Kunz's theorem:

Corollary 1.2 ([Kunz69, Corollary 2.2])**.** *If R is a regular local ring, then so is $R_\mathfrak{p}$ of R for each $\mathfrak{p} \in \operatorname{Spec} R$.*

In addition, Kunz's theorem yields the following result about excellence of $\mathbb{F}_p$-algebras.

Theorem 1.3 ([Kunz76, Theorem 2.5])**.** *Let R be a noetherian $\mathbb{F}_p$-algebra. If the Frobenius endomorphism $\varphi\colon R \to R$ is finite, then R is excellent.*

The following corollay leads to our reformulation of Kunz's theorem.

Corollary 1.4. *If R is a regular $\mathbb{F}_p$-algebra, then $R \to R_{\mathrm{perf}}$ is faithfully flat.*

Proof. It suffices to show that each $\varphi^n\colon R \to R$ is faithfully flat (cf. [SP, Tag 090N]). Since R is a regular $\mathbb{F}_p$-algebra, φ (hence φ^n) is flat by Kunz's theorem. Moreover, given a maximal ideal $\mathfrak{m} \subset R$, we have $\varphi^n(m)R \subset \mathfrak{m}^{p^n}R \subset \mathfrak{m} \neq R$. Thus we conclude that $\varphi^n\colon R \to R$ is faithfully flat. □

We will show that the converse holds, and at the same time reformulate Kunz's theorem.

Let us start with the following lemma.

Lemma 1.5 (cf. [BIM, Lemma 3.2])**.** *Let A be a perfect $\mathbb{F}_p$-algebra, and $\boldsymbol{x} = x_1, \ldots, x_n$ a sequence of elements in A.*

(1) $\sqrt{\boldsymbol{x}A} = (x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty})$.
(2) $\mathsf{fd}_A(A/\sqrt{\boldsymbol{x}A}) \le n$.

Proof. (1) Straightforward.

(2) We proceed by induction on n.

$\underline{n=1}$ Relabel $x = x_1$ for visual convenience. It suffices to check that the ideal $I := (x^{1/p^\infty}) \subset S$ is flat as an S-module (then $0 \to I \to S \to S/I \to 0$ is a flat resolution of S/I). Observe that the morphism of direct systems[1]

$$\begin{array}{ccccccc} S & \xrightarrow{x^{1-\frac{1}{p}}} & S & \xrightarrow{x^{\frac{1}{p}-\frac{1}{p^2}}} & S & \xrightarrow{x^{\frac{1}{p^2}-\frac{1}{p^3}}} & \cdots \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle x^{1/p}} & & \downarrow{\scriptstyle x^{1/p^2}} & & \\ (x) & \subset & (x^{1/p}) & \subset & (x^{1/p^2}) & \subset & \cdots \end{array}$$

induces the morphism of direct limits

$$\psi\colon \varinjlim \left(S \xrightarrow{x^{1-\frac{1}{p}}} S \xrightarrow{x^{\frac{1}{p}-\frac{1}{p^2}}} S \to \cdots \right) \to (x^{1/p^\infty}) = I.$$

The surjectivity is clear, and we can check the injectivity using the fact that B is reduced.

$\underline{n>1}$ Set $A' := A/\sqrt{x_1 A})$, and let $\boldsymbol{x}'$ be the image of the sequence $x_2, \dots, x_n$ in A'. Then A' is also perfect, and thus by the induction hypothesis, we obtain

$$\mathsf{fd}_S(S/J) \le \mathsf{fd}_{\overline{S}}(\overline{S}/\overline{J}) + 1 \le (n-1) + 1 = n,$$

(see [AF, 4.2 Corollary (b) (F)] or [SP, Tag 066K], for the first inequality). □

Now one can reformulate Kunz's theorem as the following assertion:

Proposition 1.6. *Let R be a noetherian $\mathbb{F}_p$-algebra. Then the following conditions are equivalent.*
(1) *R is regular.*
(2) *$R \to R_{\mathrm{perf}}$ is faithfully flat.*
(3) *There exists a faithfully flat ring homomorphism $R \to A$ with A perfect.*

Proof. (1) $\implies$ (2): Corollary 1.4.

(2) $\implies$ (3): Trivial.

(3) $\implies$ (1): Pick $\mathfrak{p} \in \operatorname{Spec} R$. Since $R \to A$ is faithfully flat, there exists $P \in \operatorname{Spec} A$ such that $P \cap R = \mathfrak{p}$. Then the induced local homomorphism $R_\mathfrak{p} \to A_P$ is (faithfully) flat. Thus we may assume that R, A are local and that the flat ring homomorphism $R \to A$ is local. Set $k := R/\mathfrak{m}_R$. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a s.o.p. of R ($n = \dim R$). Then lemma 1.9 yields

$$\mathsf{fd}_A(A/\sqrt{\boldsymbol{x}A}) \le n. \tag{1.1}$$

Hence, if $i > n$,

$$\begin{aligned} 0 &= \operatorname{Tor}_i^A(k \otimes_R A, A/\sqrt{\mathfrak{m}_R A}) \\ &\overset{\text{flat}}{=} \operatorname{Tor}_i^R(k, A/\sqrt{\mathfrak{m}_R A}) \\ &= \operatorname{Tor}_i^R(k,k)^{\oplus I} \end{aligned}$$

where $A/\sqrt{\boldsymbol{x}A} \cong k^{\oplus I}$ (since $A/\sqrt{\boldsymbol{x}A} = A/\sqrt{\sqrt{\boldsymbol{x}R}A} = A/\sqrt{\mathfrak{m}A}$ is a k-vector space). Since $R \to A$ is a local homomorphism, it follows that $I \neq \emptyset$, so that $\operatorname{Tor}_i^R(k,k) = 0$ for $i > n$. This means that $\operatorname{gl.dim} R = \mathsf{pd}_R k \le n < \infty$. □

Note that the essential part of the proof the finiteness of flat dimension (1.1).

[1] The notation $x^{\frac{1}{p^e}-\frac{1}{p^{e+1}}}$ makes sense because $\frac{1}{p^e} - \frac{1}{p^{e+1}} = \frac{p-1}{p^{e+1}}$.

1.2. **Mixed characteristic.** We insert here a brief review of the definition and some properties of perfectoid rings (cf. [BMS1]).

Definition 1.7 ([BIM, Definition 3.5])**.** We say that a ring A is *perfectoid* if it satisfies the following conditions.

(1) A is p-adically complete.
(2) The $\mathbb{F}_p$-algebra A/pA is semiperfect.
(3) The kernel of Fontaine's map $\theta\colon W(A^\flat) \to A$ is principal.
(4) There exist $\pi \in A$ and $u \in A^\times$ such that $\pi^p = pu$.

For the equivalence of this definition and other characterizations, see [BMS1, Lemma 3.9, Proposition 3.10]. If A is perfectoid, then the following hold:

- Fontaine's map $\theta\colon W(A^\flat) \to A$ is surjective (see [BMS1, Lemma 3.9]).
- An element $\xi = (\xi_0, \xi_1, \dots) \in \ker\theta$ generates $\ker\theta$ if and only if ξ is distinguished, i.e., $\xi_1 \in (A^\flat)^\times$ (see [BMS1, Remark 3.11]).

Note also that an arbitrary product of perfectoid rings is perfectoid ([BIM, Example 3.8 (8)]). Indeed, the condition (3) in Definition 1.7 is satisfied because the functor $A \mapsto W(A^\flat)$ commutes with products. The other conditions are obvious.

We now turn to generalizing Kunz's theorem to mixed characteristic. The most important features of perfectoid rings are the following.

Lemma 1.8 ([BIM, Lemma 3.7])**.** *Let A be a perfectoid ring.*

(1) *The $\mathbb{F}_p$-algebra $\overline{A} := A/\sqrt{pA}$ is perfect.*
(2) *The ideal $\sqrt{pA} \subset A$ is a flat A-module.*

Proof. (1) We first show that the element π appearing in Definition 1.7 can be assumed to admit a compatible system of p-power roots $\{\pi^{1/p^n}\}_{n\geq 1}$. Let $\xi = (\xi_0, \xi_1, \dots) \in W(A^\flat)$ be a generator of $\ker\theta = (\xi)$, and set $\pi \in A$ to be the image of $[\xi_0]$. Then π satisfies the condition (4) in Definition 1.7 (here we use that ξ is distinguished) and admits a compatible system of p-power roots, namely the images of $[\xi_0^{1/p^{n+1}}]$.

Since $(p) = (\pi^p)$, one has $\sqrt{pA} = (p^{1/p^\infty}) = (\pi^{1/p^\infty})$, and so it suffices to show that $A/(\pi^{1/p^\infty})$ is perfect. The isomorphism $W(A^\flat)/(\xi) \xrightarrow{\sim} A$ and the definition of our π yield

$$A/(\pi^{1/p^\infty}) \cong W(A^\flat)/(\xi, [\xi_0^{1/p^\infty}]) \cong \frac{W(A^\flat)/(p)}{(\xi, [\xi_0^{1/p^\infty}])/(p)} \cong A^\flat/\left(\xi_0^{1/p^\infty}\right) \overset{\text{Lemma 1.5 (1)}}{=} A^\flat/\sqrt{(\xi_0)}.$$

This ring is perfect since $A^\flat$ is perfect.

(2) We prove only in the case where A is p-torsion free. (The general case is difficult.) We check that $\mathsf{fd}_A(\overline{A}) \leq 1$; this is equivalent to showing that $\sqrt{pA}$ is flat. Since A is p-torsion free and $\pi^p = pu$ for some $u \in A^\times$, we can see that each π^{1/p^e} is a non-zero-divisor of A. Thus each $\varpi^{1/p^e}A$ is isomorphic to A, hence is free. The directed union (π^{1/p^∞}) is also flat. □

By this lemma, we deduce the desired finiteness of flat dimension. For the sake of the later applications, we give this result in more general setting. We say that a ring A is *(positive characteristic p and) perfect modulo a flat ideal* if there exists an ideal $I \subset A$ containing p such that A/I is perfect. Lemma 1.8 shows that any perfectoid is perfect modulo a flat ideal.

Lemma 1.9. *Let A be a ring that is perfect modulo a flat ideal I, and set $\overline{A} := A/I$. If $J \subset A$ is an ideal containing I such that $J\overline{A} = \sqrt{\boldsymbol{x}\overline{A}}$ for some sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\overline{A}$. Then $\mathsf{fd}_A(A/J) \leq n+1$.*

Proof. $\mathsf{fd}_A(A/J) \leq \mathsf{fd}_{\overline{A}}(\overline{A}/J\overline{A}) + \mathsf{fd}_A \overline{A} \overset{\text{Lemma 1.8}}{\leq} n+1.$ □

Now we can prove a mixed characteristic generalization of Kunz's theorem.

Theorem 1.10 ([BIM, Theorem 4.7])**.** *Let R be a noetherian ring with $p \in \operatorname{rad} R$. Then the following conditions are equivalent.*

(1) *R is regular.*
(2) *There exists a faithfully flat ring homomorphism $R \to A$ with A perfectoid.*

Proof. (2) ⇒ (1): Because of lemma 1.8 and lemma 1.9, the argument similar to that in Proposition 1.6 works.

(1) $\Rightarrow$ (2): Assume that R is regular with $p \in \operatorname{rad} R$. We must construct a faithfully flat ring homomorphism $R \to A$ with perfectoid.

(Step 1): Reduction to the case where R is complete local. Assume that, for each $\mathfrak{m} \in \operatorname{Max} R$, we obtain a faithfully flat ring homomorphism $\widehat{R}_\mathfrak{m} \to A(\mathfrak{m})$ with $A(\mathfrak{m})$ perfectoid. Consider the resulting ring homomorphism

$$R \to \prod_{\mathfrak{m} \in \operatorname{Max} R} \widehat{R}_\mathfrak{m} \to \prod_{\mathfrak{m} \in \operatorname{Max} R} A(\mathfrak{m}).$$

As R is noetherian, an arbitrary product of flat R-modules is flat ([岩永-佐藤, 命題 8-2-7]), so the above map is flat. Moreover, it is also faithfully flat: the induced map $\operatorname{Spec}\left(\prod_{\mathfrak{m} \in \operatorname{Max} R} A(\mathfrak{m})\right) \to \operatorname{Spec} R$ is open (by flatness), and thus its image is generization-closed. Moreover, the image contains all closed points by construction. As a product of perfectoid rings is perfectoid, we have constructed the desired covers.

(Step 2): Reduction to the case where R is a domain. Since R is regular, we can write $R = \prod_{i \in I} R_i$ with R_i regular domain and I finite ([BH, Corollary 2.2.20]). If we obtain a faithfully flat ring homomorphism $R_i \to A_i$ with A_i perfectoid, for each $i \in I$, then the product $R = \prod_{i \in I} R_i \to \prod_{i \in I} A_i =: A$ is a faithfully flat ring homomorphism with A perfectoid (faithfully flatness follows since I is finite).

(Step 3): Finish. Since we have seen the positive characteristic case in Proposition 1.6, it remains the case of mixed characteristic $(0, p)$ (we note that $p \in \operatorname{rad} R$). By [BouAC2, IX, App., Theorem 1, Corolally], there exists a *gonflement* $R \to S$ such that the residue field of S is an algebraically closure of $R/\mathfrak{m}_R$. Then $R \to S$ is faithfully flat ([BouAC2, IX, App., Proposition 2, b)]) and S is also regular ([BouAC2, IX, App., Proposition 2, Corollary)]). Thus we may replace R by S, hence may assume that $R/\mathfrak{m}_R$ is perfect. Then, by Cohen's structure theorem,

$$R = \begin{cases} W(k)[[X_2, \dots, X_d]] & \text{if } R \text{ is unramified,} \\ W(k)[[X_1, \dots, X_d]]/(p - f) & \text{if } R \text{ is ramified.} \end{cases}$$

where $f \in (x_1, \dots, x_d)^2 \setminus (p)$. We take

$$A := \begin{cases} \left(W(k)[p^{1/p^\infty}][[X_2^{1/p^\infty}, \dots, X_d^{1/p^\infty}]]\right)_p^\wedge & \text{if } R \text{ is unramified,} \\ \left(W(k)[[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]]/(p - f)\right)_p^\wedge & \text{if } R \text{ is ramified.} \end{cases}$$

Indeed, A is perfectoid and $R \to A$ is faithfully flat:

(perfectoid): The unramified case follows as in the case $k = \mathbb{F}_p$ (hence $W(k) = \mathbb{Z}_p$). The ramified case is due to Shimomoto [Shi16, Proposition 4.9].

(faithfully flat): Observe that

$$S := \begin{cases} W(k)[p^{1/p^\infty}][[X_2^{1/p^\infty}, \dots, X_d^{1/p^\infty}]] = \bigcup_{n > 0} W(k)[p^{1/p^n}][[X_2^{1/p^n}, \dots, X_d^{1/p^n}]], \\ W(k)[[X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}]]/(p - f) = \bigcup_{n > 0} W(k)[[X_1^{1/p^n}, \dots, X_d^{1/p^n}]]/(p - f). \end{cases}$$

In the ramified case, $R \to A$ is faithfully flat by [Bha18, Proposition 5.12] and the fact that $R/pR \to S/pS \xrightarrow{\sim} A/pA$ is faithfully flat. ∎

2. Criteria for finite flat dimension

2.1. **Local cohomology.** Local cohomology is an important tool in homological algebra because of the Grothendieck (non-)vanishing results related to the depth and dimension of a finitely generated module over a Noetherian local ring (cf. [BH, Theorem 3.5.7]). Let us briefly the definition of local cohomology.

Let R be a ring, and $\mathfrak{a} \subset R$ an ideal. For an R-module M, the $\mathfrak{a}$-torsion submodule of M is defined by

$$\Gamma_{\mathfrak{a}}(M) \overset{\text{def}}{=} \bigcup_{q \geq 0} 0 :_M \mathfrak{a}^q \cong \varinjlim_k \operatorname{Hom}_R(R/\mathfrak{a}^q, M).$$

Then $\Gamma_{\mathfrak{a}}(-)$ defines a left exact functor from the category $\mathsf{Mod}\, R$ of R-modules into itself. The right derived functors of $\Gamma_{\mathfrak{a}}(-)$ is called the *local cohomology functors*, denoted by $H^i_{\mathfrak{a}}(-)$.

Remark 2.1. On the other hand, the completion functor $\Lambda^{\mathfrak{a}}(-)$ in commutative algebra, which is given by

$$\Lambda^{\mathfrak{a}}(M) \overset{\text{def}}{=} \varprojlim_q (R/\mathfrak{a}^q \otimes_R M),$$

is formally dual to the functor $\Gamma_{\mathfrak{a}}(-)$. However, it is neither left nor right exact in general. In order to study the left derived functors $L_i\Lambda^{\mathfrak{a}}$, Greenless and May [GM92] introduced the notion of a "proregular sequence," which we will introduce in 3.1.

For the sake of our applications in this section, we just remark here the following result:

Theorem 2.2. *Let R be a noetherian ring, and M an R-module. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements in R, and set $\mathfrak{a} := (\boldsymbol{x})$. Then*

$$H^i_{\mathfrak{a}}(M) \cong \check{H}^i(\boldsymbol{x}; M) := H^i(\check{C}(\boldsymbol{x}) \otimes_R M)$$

For the proof, see [BH, Theorem 3.5.6]. Note that this implies the following result, which

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M an R-module. Then*

$$s(M) := \sup\{t \mid H^t_{\mathfrak{m}}(\check{M})\} \neq 0\} \leq \dim R.$$

where $\check{M} := \operatorname{Hom}_R(M, E_R(k))$.

Proof. If $\boldsymbol{x}$ is a system of parameters of R, then $\check{C}^i(\boldsymbol{x}) = 0$ for $i > \dim R$, and thus $H^i_{\mathfrak{m}}(M) \overset{\text{Theorem 2.2}}{\cong} H^i(\check{C}(\boldsymbol{x}) \otimes_R M) = 0$ for $i > \dim R$, which confirms the assertion. ☐

In addition, we need the following fact, which concerns a rigidity.

Fact 2.4 ([CIM19, Proposition 3.3])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and M an R-module. Set $s(M)$ be as in Theorem 2.2. If $\operatorname{Tor}^R_i(M, k) = 0$ for some $i \geq s(M)$, then $\operatorname{Tor}^R_j(M, k) = 0$ for all $j \geq i$.*

2.2. **Results in** [BIM]**.**

Theorem 2.5 ([BIM, Theorem 2.1])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, and S an R-algebra containing an ideal J with $\mathfrak{m}S \subseteq J$ and $\mathsf{fd}_S(S/J) < \infty$. Let $d \geq \mathsf{fd}_S(S/J)$ be an integer, and U an S-module with $JU \neq U$. Let M be an R-module, and $s \geq 0$ an integer. If $\operatorname{Tor}^R_j(U, M) = 0$ for $j \geq s$, then $\operatorname{Tor}^R_j(k, M) = 0$ for $j \geq s + d$.*

Theorem 2.6 ([BIM, Theorem 4.1, Remark 4.3])**.** *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring with $\operatorname{char}(k) = p$, and A an R-algebra with $\mathfrak{m}A \neq A$. Assume that $R \to A$ factors through an R-algebra S that is perfect modulo a flat ideal. Let M be an R-module. If $\operatorname{Tor}^R_j(A, M) = 0$ for $j \gg 0$, then $\operatorname{Tor}^R_j(k, M) = 0$ for $j \gg 0$.*

Proof of Theorem 2.6. Apply Theorem 2.5 when the case:
- $J = \sqrt{\mathfrak{m}S} = \sqrt{\boldsymbol{x}S}$, where $\boldsymbol{x} = x_1, \ldots, x_n$ is a s.o.p. of R.
- $d := n + 1$.
- $U := A$.

Note that if follows from lemma 1.9 that $\mathsf{fd}_S(S/J) \leq n + 1 = d$. ☐

3. Applications

3.1. (Weakly) Proregular sequences. We refer to [Sch03] and [SS].

At the early times of local cohomology, Grothendieck proved that the local cohomology of a module over a Noetherian ring may be computed with a Čech complex (see Theorem 2.2): In [AJL97, Lemma (3.1.1)], Lipman et al. generalized the result Theorem 2.2 to the case of $\boldsymbol{x}$ a proregular sequence in an arbitrary commutative ring. It turned out that this is not correct (pointed out by Schenzel), and Lipman suggested the notion of a "weakly proregular sequence." Schenzel proved that Čech cohomology coincides with local cohomology if and only if the ideal is generated by a weakly proregular sequence, see [Sch03, Theorem 3.2], which was originated by [AJL97] and [GM92]

Observation 3.1. Let R be a ring, and $\boldsymbol{x} = x_1, \dots, x_d$ a sequence in R. If $\boldsymbol{x}$ is an R-regular sequence, then for each $n \geq 1$, so is $x_1^n, \dots, x_d^n$ is again an R-regular sequence (see [Mat2, Theorem 16.1]) and so,

$$\frac{(x_1^n, \dots, x_{i-1}^n) :_R x_i^n}{(x_1^n, \dots, x_{i-1}^n)} = 0.$$

Due to this observation, we see that proregular sequences, defined as the following, is indeed a generalization of regular sequences.

Definition 3.2. Let R be a ring, and $\boldsymbol{x} = x_1, \dots, x_d$ a sequence in R.

$$\frac{(x_1^m, \dots, x_{i-1}^m) :_R x_i^m}{(x_1^m, \dots, x_{i-1}^m)} \xrightarrow{x_i^{m-n}} \frac{(x_1^n, \dots, x_{i-1}^n) :_R x_i^n}{(x_1^n, \dots, x_{i-1}^n)}$$

is zero, i.e.,

$$(x_1^m, \dots, x_{i-1}^m) :_R x_i^m \subset (x_1^n, \dots, x_{i-1}^n) :_R x_i^{m-n}.$$

The following notion will be particularly important.

Definition 3.3 ([Gro, Definition, p. 23])**.** Let R be a ring. We say that an inverse system $\{X_n\}_{n \geq 0}$ of R-modules is *prozero* (or *essentially zero, essentially null*) if for any $n \geq 0$, there exists $m \geq n$ such that the transition map $X_m \to X_n$ is zero.

If the inverse system $\{X_n\}_{n \geq 0}$ is prozero, then $\varprojlim_n X_n = 0$. The converse is false (e.g., consider the inverse system $\{p^n \mathbb{Z}_p\}_{n \geq 0}$ of $\mathbb{Z}_p$-modules).

Definition 3.4. Let R be a ring, and $\boldsymbol{x} = x_1, \dots, x_d$ a sequence in R. We say that $\boldsymbol{x}$ is *weakly proregular* if for any $1 \leq i \leq d$, the inverse system $\{H_i(x_1^n, \dots, x_d^n)\}_{n \geq 0}$ is prozero.

Remark 3.5.
(1) The following implications hold:

$$\text{'regular'} \Longrightarrow \text{'proregular'} \Longrightarrow \text{'weakly proregular'}$$

The first implication is what we have seen. For the proof of the second implication, we refer to [Sch03, Lemma 2.7], or [SS, Lemma A.2.3] for modules, or [Sch21, Theorem 3.4] for homological approach.

(2) If R is noetherian, then any sequence $\boldsymbol{x} = x_1, \dots, x_d$ in R is proregular. To verify this, we only have to remark that for a fixed integer $n \geq 1$, the increasing sequence $\{(x_1^n, \dots, x_{i-1}^n :_R x_i^{m-n}\}_{m \geq n}$ of ideals of R is stationary.

Example 3.6. For a length one sequence x, we can see that the following conditions are equivalent.
(1) x is poregular.
(2) x is weakly proregular.
(3) The increasing sequence $0 :_R x \subset 0 :_R x^2 \subset 0 :_R x^3 \subset \cdots$ is stationary.
(4) There exists an integer $n \geq 1$ such that $x^n R = 0$.

When these conditions are satisfied, we also say that R *is of xR-bounded torsion.*

Example 3.7. Let $R := \prod_{n>0} \mathbb{Z}/2^n\mathbb{Z}$, $x := (2, 2, \dots) \in R$. Then

(1) x is neither weakly proregular nor proregular: given $m > 0$, $a := (\overbrace{\ [\ }\ 0, \dots, 0]m, 1, 1, \dots)$ satisfies $ax^{m+1} = 0$ but $ax^m \neq 0$, hence $0 :_S x^m \subsetneq 0 :_S x^{m+1}$. Thus S is not of bounded xS-torsion.
(2) $x, 1$ is not pro-regular (by (1)).
(3) $1, x$ is pro-regular (hence weakly pro-regular): Since $0 :_S 1 = 0 :_S 1^2 = \cdots S$, it follows that S is of bounded $1S$-torsion. For $n > 0$, $1^n S :_S x^n = S = 1^n S :_S x^{n-n}$, and so we may take $m := n$.

By (2) and (3), a pre-regular sequence is not permutable without any additional assumption.

We care about these notions because of the following observation.

Lemma 3.8 ([BIM, Lemma 4.10])**.** *Let R be a noetherian ring, and S an R-algebra. If an ideal $\mathfrak{a} \subset R$ is generated, up to radical, by a sequence whose image in S is weakly proregular, then $H^i_{\mathfrak{a}}(I) = 0$ for $i \geq 1$ and any injective S-module I.*

Proof. By hypothesis, there exists a sequence $\boldsymbol{x}$ in R such that $\sqrt{(\boldsymbol{x})} = \sqrt{\mathfrak{a}}$ and $\boldsymbol{x}S$, the image of the sequence $\boldsymbol{x}$ in S, is weakly proregular. Then we have:

$$H^i_{\mathfrak{a}}(I) \overset{\text{Pf. of [BH, 3.5.6]}}{=} H_i(\check{C}(\boldsymbol{x}) \otimes_R I) = H_i((\check{C}(\boldsymbol{x}) \otimes_R S) \otimes_S I) = H_i(\check{C}(\boldsymbol{x}S) \otimes_S I) \overset{\text{[Sch03, Thm. 3.2]}}{=} 0.$$

□

3.2. **Application to R^+ and R_{perf}.** Given a domain R, its *absolute integral closure* R^+ is the its integral closure in an algebraic closure of it field of fractions. When R is of positive characteristic, R^+ contains a subalgebra isomorphic to R_{perf}.

When R has mixed characteristic, with residual characteristic p, the ideal $(p^{1/p^\infty})R^+$ is flat, and the quotient ring $R^+/(p^{1/p^\infty})R^+$ is of characteristic p and perfect.

Their most important features for the aim are the following.

Proposition 3.9 ([BIM, Proposition 4.11])**.** *Let R be an excellent local domain, and $\boldsymbol{x}$ a system of parameters of R.*

(1) *If R has positive characteristic, then $\boldsymbol{x}$ is weakly proregular in R_{perf} and in R^+.*
(2) *If R has mixed characteristic and $\dim R \leq 3$, then $\boldsymbol{x}$ is weakly proregular in R^+.*

Sketch. The assertions for R^+ hold if R^+ is a balanced big Cohen-Macaulay algebra (since a regular sequence is a weakly proregular sequence). This is indeed the case where:

- R has positive characteristic, as proved by Hochster and Huneke [HH92, Theorem 1.1].
- R has mixed characteristic and $\dim R \leq 2$, as well-known.

It remains to the case where R has mixed characteristic and $\dim R = 3$. But, even in this case, we have a useful result of by Heitmann's [Hei05, Theorem 0.1] (see also [Hei02]), and we can prove the assertion.

We omit the proof for R_{perf}. □

Corollary 3.10 ([BIM, Corollary 4.12])**.** *Let $(R, \mathfrak{m}, k)$ be an excellent local domain, and $(-)^\vee := \mathrm{Hom}_R(-, E_R(k))$ the Matlis dual.*

(1) *If R has positive characteristic, then $H^i_{\mathfrak{m}}((R^+)^\vee) = H^i_{\mathfrak{m}}((R_{\mathrm{perf}})^\vee) = 0$ for $i \geq 1$.*
(2) *If R has mixed characteristic and $\dim R \leq 3$, then $H^i_{\mathfrak{m}}((R^+)^\vee) = 0$ for $i \geq 1$.*

Proof. By adjunction, the R^+-module $(R^+)^\vee$ and the R_{perf}-module $(R_{\mathrm{perf}})^\vee$ are injective. Thus the desired result follows from Proposition 3.9 and lemma 3.8. □

Theorem 3.11 ([BIM, Theorem 4.13])**.** *Let $(R, \mathfrak{m}, k)$ be an excellent local domain. Then R is regular if any one of the following conditions.*

(1) *R has positive characteristic and $\mathrm{Tor}^R_i(R_{\mathrm{perf}}, k) = 0$ for some $i \geq 1$;*
(2) *R has positive characteristic and $\mathrm{Tor}^R_i(R^+, k) = 0$ for some $i \geq 1$;*
(3) *R has mixed characteristic, $\dim R \leq 3$, and $\mathrm{Tor}^R_i(R^+, k) = 0$ for some $i \geq 1$.*

Remark 3.12. Aberbach and Li [AL08, Corollary 3.5] have proved parts (1) and (2), using different methods.

Proof of Theorem 3.11. In all cases, it follows from Corollary Corollary 3.10 and Fact 2.4 that $\mathrm{Tor}^R_j(R^+, k)$, respectively, $\mathrm{Tor}^R_j(R_{\mathrm{perf}}, k)$, is zero for each $j \geq i$. When R has positive characteristic, R^+ contains R_{perf}; in mixed characteristic, R^+ is perfect modulo a flat ideal. Therefore, in either case Theorem 2.6 implies $\mathrm{Tor}^R_j(k, k) = 0$ for $j \gg 0$ as desired. □

Here is a question suggested by part (3) above: If $(R, \mathfrak{m}, k)$ is a noetherian local domain of characteristic 0 and $\mathrm{Tor}^R_i(R^+, k) = 0$ for some $i \geq 1$, then is R regular?

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