Frobenius maps on mixed characteristic rings via prismatic cohomology

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Kunz's Theorem

Kunz's Theorem (Kunz)[1]

Let R be a Noetherian local ring of characteristic $p > 0$. TFAE.

1. R is regular.
2. The Frobenius map $F\colon R \to R$ is (faithfully) flat.
3. The canonical map $R \to R_{\mathrm{perf}} := \operatorname{colim}_F R$ is (faithfully) flat.

Can we generalize Kunz's theorem to mixed characteristic rings?

Observation

- R_{perf} is perfect of characteristic p, namely, the Frobenius map $F_{R_{\mathrm{perf}}}$ is an isomorphism.
- A ring A of characteristic p is perfectoid if and only if A is perfect.

[1] American Journal of Mathematics, 91 (1969) 772–784.

p-adic Kunz's Theorem

p-adic Kunz's Theorem (Bhatt–Iyengar–Ma)[2]

Let R be a Noetherian ring with $p \in \mathrm{Jac}(R)$. TFAE.

1. R is regular.
2. There exists a perfectoid ring A and a faithfully flat map $R \to A$.

Definition

A p-adically complete ring A is *perfectoid* if

1. there exists an element $\pi \in A$ such that $p \in \pi^p A$,
2. the Frobenius map $F\colon A/pA \to A/pA$ is surjective, and
3. the kernel $\ker(\theta)$ of the Fontaine's theta map $\theta\colon W(A^\flat) \to A$ is principal.

If A is p-torsion-free and satisfies (1) and (2), then (3) is equivalent to that $A/\pi A \xrightarrow{a \mapsto a^p} A/\pi^p A$ is an isomorphism.

[2] Communications in Algebra, 47(6) (2019) 2367–2383.

What is the "mixed characteristic Frobenius map"?

What about the "mixed characteristic Frobenius map"?

Observation (cf. Bhatt)[3]

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring with $\mathrm{char}(k) = p > 0$ (and some extra conditions). The "mixed characteristic Frobenius map" is the Frobenius lift of the *prismatic cohomology* $\Delta_{R/A}$ of $A/I \to R$ for some *prism* (A, I).

Definition

A *(bounded) prism* (A, I) is a pair of a δ-ring A and a locally free of rank 1 ideal I of A such that

1. A is (p, I)-adically complete,
2. $p \in I + \varphi_A(I)A$, and
3. A/I has bounded p^∞-torsion, i.e., $A/I[p^\infty] = A/I[p^N]$ for some $N \geq 0$.

[3] Proceedings of the International Congress of Mathematicians, 2, (2022), 712–748.

Prisms and Prismatic Cohomology

We briefly recall the notion of prismatic cohomology.

Prismatic Cohomology (Bhatt–Scholze)[4]

Let (A, I) be a bounded prism (e.g., $(\mathbb{Z}_p[|T_1, \ldots, T_n|], (p - f))$).
For a given (animated) A/I-algebra $A/I \to R$, we can define the *prismatic cohomology* $\mathbb{\Delta}_{R/A}$ of $A/I \to R$. Naively, this is a commutative algebra object in the derived category $D(A)$ of A-modules:

$$\mathbb{\Delta}_{R/A} \in D(A).$$

Moreover, $\mathbb{\Delta}_{R/A}$ is equipped with a φ_A-semilinear endomorphism $\varphi\colon \mathbb{\Delta}_{R/A} \to \mathbb{\Delta}_{R/A}$, which is called the *Frobenius lift* of $\mathbb{\Delta}_{R/A}$.

An important property of prismatic cohomology is that it specializes to a lot of integral p-adic cohomology theories such as crystalline cohomology, de Rham cohomology, p-adic étale cohomology, and so on.

[4] Annals of Mathematics, 196(3) (2022) 1135–1275.

Setting

Construction

Let (A, I) be a bounded prism and let $J := (I, f_1, \ldots, f_r) \subseteq A$. We set

1. an A/I-algebra $R := A/J$ and
2. an animated A/I-algebra $R^{an} := R^{an}(f_1, \ldots, f_r) := (A/I) \otimes^L_{\mathbb{Z}[\underline{X}]} \mathbb{Z}$, where $A/I \xleftarrow{f_i \leftarrow\!\shortmid X_i} \mathbb{Z}[\underline{X}] = \mathbb{Z}[X_1, \ldots, X_r] \xrightarrow{X_i \mapsto 0} \mathbb{Z}$.

If $f_1, \ldots, f_r$ is a regular sequence on A/I, then $R^{an} \xrightarrow{\cong} R$.

Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring and let $n := \mathrm{emdim}(R)$. By Cohen's structure theorem, we can take a bounded prism $(A, I) = (\mathbb{Z}_p[|T_1, \ldots, T_n|], (p - f))$ for some $f \in (T_1, \ldots, T_n)$ and a surjection $A/I \xrightarrow{\pi} R$. Fix a finite generator $f_1, \ldots, f_r$ of $\ker(\pi)$ and take $R^{an} = R^{an}(f_1, \ldots, f_r)$.

Main Theorem

Prismatic Kunz's Theorem (Ishizuka–Nakazato)[5]

Under the above setting, TFAE.

1. R is regular.
2. The Frobenius lift $\varphi\colon {\mathbb{\Delta}}_{R^{an}/A} \to {\mathbb{\Delta}}_{R^{an}/A}$ is faithfully flat.

Remark

- If R is complete intersection, we can assume that $f_1, \dots, f_r$ is a regular sequence. In this case R is regular if and only if $\varphi\colon {\mathbb{\Delta}}_{R/A} \to {\mathbb{\Delta}}_{R/A}$ is faithfully flat.
- The proof uses the p-adic Kunz's theorem by Bhatt–Iyengar–Ma and the following lemma.

[5] preprint, arXiv:2402.06207.

Essential Lemma

To prove the main theorem, we need the following general lemma.

Lemma (Ishizuka–Nakazato)

Let (A, I) be a bounded prism and let $J := (I, f_1, \ldots, f_r) \subseteq A$. Then $R^{an} \to \overline{\mathbb{\Delta}}_{R^{an}/A} := \mathbb{\Delta}_{R^{an}/A} \otimes^L_A (A/I)$ is p-completely faithfully flat, namely, $R/pR \otimes^L_{R^{an}} \overline{\mathbb{\Delta}}_{R^{an}/A}$ is concentrated in degree 0 and is a faithfully flat R/pR-module.
If R is Noetherian, $R \to \pi_0(\overline{\mathbb{\Delta}}_{R^{an}/A})$ is faithfully flat.

Sketch (main theorem)

If $\varphi\colon \mathbb{\Delta}_{R^{an}/A} \to \mathbb{\Delta}_{R^{an}/A}$ is faithfully flat, then the composition

$$R^{an} \to \overline{\mathbb{\Delta}}_{R^{an}/A} \to \overline{\mathbb{\Delta}}_{R^{an}/A,\infty} := ((\operatorname*{colim}_{\varphi} \mathbb{\Delta}_{R^{an}/A}) \otimes^L_A A/I)^{\wedge p}$$

is p-completely faithfully flat. Since $\mathbb{\Delta}_{R^{an}/A}$ is connective, the target is concentrated in degree 0. By p-adic Kunz's theorem, R is regular.