# ABSOLUTE INTEGRAL CLOSURE

RYO ISHIZUKA

Abstract. This is a note for the 18th summer school on Commutative algebra at the Tokyo Institute of Technology to introduce some research about absolute integral closure. The conference webpage is Here (in Japanese).

Of course, everything written here is not the result of the authorship, and any errors are the responsibility of the author. If you find a mistake, please let me know.

## Contents

1. Introduction 2
2. Cohen-Macaulayness of Absolute Integral Closure 3
2.1. Ind-CM objects 3
2.2. Positive chcaracteristic case 3
3. Proof strategy 5
3.1. Notations 5
3.2. Alterations over $X$ 5
3.3. Key lemmas 6
3.4. $(*)_{CM}$ condition 7
3.5. A Reduction step 7
3.6. Some lemmas 8
3.7. Sketch of proof 9
References 11

*Date*: August 29, 2023.

## 1. Introduction

**Definition 1.1.** Let $R$ be a domain and let $K$ be the fraction field of $R$. Fix an algebraic closure $\overline{K}$ of $K$. The *absolute integral closure* $R^+$ is the integral closure of $R$ in $\overline{K}$.

**Theorem 1.2** ([HH92; HL07])**.** *Let $(R, \mathfrak{m})$ be an excellent Noetherian local domain of characteristic $p$. Then $H^i_{\mathfrak{m}}(R^+)$ vanishes for $i < \dim(R)$ and $R^+$ is a big Cohen-Macaulay $R$-algebra. In [Quy16], this is true for any Noetherian local domain of characteristic $p$ which is an image of a Cohen-Macaulay local ring.*

**Theorem 1.3** ([Bha21, §5])**.** *Let $(R, \mathfrak{m})$ be an excellent $p$-Zariskian Noetherian local domain. Then we have the following:*

*(1) $H^i_{\mathfrak{m}}(R^+/(p))$ vanishes for $i < \dim(R/(p))$.*
*(2) $H^i_{\mathfrak{m}}(R^+)$ vanishes for $i < \dim(R)$.*
*(3) $\widehat{R^+}$ is a big Cohen-Macaulay $R$-algebra.*
*(4) If $R$ is splinter, $R$ is CM.*
*(5) If $R$ is regular, $R \to \widehat{R^+}$ is faithfully flat.*

## 2. Cohen-Macaulayness of Absolute Integral Closure

### 2.1. Ind-CM objects.

DefIndCM

**Definition 2.1** (Ind-CM objects [Bha21, Definition 2.10])**.** For a finite dimensional scheme $X$, an ind-object $\{M_k\}$ in $D_{qc}(X)$ is *ind-CM* if the following holds;

- For all $x \in X$, the ind-object $\{H^i_x(M_{k,x})\}$ of $\mathcal{O}_{X,x}$-modules is 0 for any $i < \dim(\mathcal{O}_{X,x})$.

Ind-CMInd-FiniteNoetherian

**Lemma 2.2** (Ind-CM object on non-closed points is ind-(finite length) object: a Noetherian case [Bha21, Lemma 2.16] and [HL07, Theorem 2.1])**.** *Let $X$ be a biequidimensional Noetherian scheme that admits a normalized dualizing complex $\omega^\bullet_X$. Let $\{M_k\}$ be an ind-object in $D^b_{coh}(X)$ which is ind-CM after restriction to any* non-closed point *of $X$.*[1]

*Then, for each closed point $x \in X$ and each $i < \dim(\mathcal{O}_{X,x})$, the ind-object $\{H^i_x(M_{k,x})\}$ of $\mathcal{O}_{X,x}$-modules is isomorphic to an ind-object $\{\mathcal{I}^{(i,x)}_k\}$, where every $\mathcal{I}^{(i,x)}_k$ is finite length $\mathcal{O}_{X,x}$-module contained in some $H^i_x(M_{k,x})$.*[2]

### 2.2. Positive chcaracteristic case.

The following lemma is called "equational lemma" in [HL07].

EquationalLemCharp

**Lemma 2.3** ([HL07, Lemma 2.2])**.** *Let $R$ be a Noetherian domain of positive characteristic $p$ and let $I$ be an ideal of $R$. Let $K$ be the fraction field of $R$ and let $\overline{K}$ be the algebraic closure of $K$. Fix an element $\alpha \in H^i_I(R)$. Because of $\mathrm{char}(R) = p$, $H^i_I(R)$ has the Frobenius action $F$ and we denote $F^e(\alpha)$ by $\alpha^{p^e}$. We assume that the elements $\alpha, \alpha^p, \alpha^{p^2}, \dots, \in H^i_I(R)$ belong to a finitely generated $R$-submodule of $H^i_I(R)$.*

*Then there exists a finite extension $R \hookrightarrow S$ contained in $\overline{K}$ such that $H^i_I(R) \to H^i_I(S)$ induced by the extension sends $\alpha$ to 0.*

By using Lemma 2.2 and Lemma 2.3, we can show the following:

**Theorem 2.4** ([HL07, Theorem 2.1])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local domain of characteristic $p$ and let $K$ be the fraction field of $R$ with the algebraic closure $\overline{K}$. Assume that $R$ has a dualizing complex. Then for any finite $R$-algebra $R'$ contained in $\overline{K}$, there exist a finite extension $R' \hookrightarrow S$ contained in $\overline{K}$ such that, for any $i < \dim(R)$, the map $H^i_{\mathfrak{m}}(R') \to H^i_{\mathfrak{m}}(S)$ induced by the extension is the 0 map.*

*More precisely, the ind-object $\{S\}_{R \subseteq S \subseteq R^+}$ where $S$ runs through every finite $R$-algebra contained in $R^+$ is ind-CM in the sense of Definition 2.1. That is, for every prime ideal $\mathfrak{p} \subset R$, the ind-object $\{H^i_{\mathfrak{p}}(S_{\mathfrak{p}})\}$ of $R_{\mathfrak{p}}$-modules is 0 for any $i < \dim(R_{\mathfrak{p}})$.*

[1] That is, for any *non-closed point* $x \in X$, the ind-object $\{H^i_x(M_{k,x})\}$ of $\mathcal{O}_{X,x}$-modules is 0 for $i < \dim(\mathcal{O}_{X,x})$

[2] More precisely, for any $H^i_x(M_{k,x})$, there exists $k' \geq k$ such that the transition map factors through $H^i_x(M_{k,x}) \twoheadrightarrow \mathcal{I}^{(i,x)}_k \hookrightarrow H^i_x(M_{k',x})$.

*Proof.* We use induction on $d \coloneqq \dim(R)$. If $d = 0$, $R$ is a field and thus has nothing to prove. We assume that $d > 0$ and the ind-CM property for all smaller dimensions.

Applying Lemma 2.2 (3) for $X = \mathrm{Spec}(R)$ and $\{M_k\} = \{S\}_{R \subseteq S \subseteq R^+}$, we can show that, for each $i < \dim(R) = d$, the ind-object $\{H^i_{\mathfrak{m}}(S)\}$ of $R$-modules is isomorphic to an ind-object $\{\mathcal{I}^{(i)}\}$ of $R$-modules such that $\mathcal{I}^{(i)}$ is finite length $R$-module. More precisely, for each $S$, there exists a finite extension $S \hookrightarrow S'$ contained in $R^+$ such that $H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S')$ factors through $H^i_{\mathfrak{m}}(S) \twoheadrightarrow \mathcal{I}^{(i)} \hookrightarrow H^i_{\mathfrak{m}}(S')$ for some $\mathcal{I}^{(i)}$.

Let $\alpha_1, \dots, \alpha_s \in H^i_{\mathfrak{m}}(S')$ be a generator of $\mathcal{I}^{(i)}$. Since the Frobenius maps over $S$ and $S'$ are compatible with the extension $S \hookrightarrow S'$, the Frobenius action on each local cohomology commutes with the canonical map $H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S')$. Then the set of elements $\{\alpha_i^{p^j} \mid 1 \leq i \leq s, 0 \leq j\} \subseteq H^i_{\mathfrak{m}}(S')$ is contained in $\mathcal{I}^{(i)}$. By the above equational lemma Lemma 2.3, there exists a finite extension $S' \hookrightarrow S''$ contained in $R^+$ such that $H^i_{\mathfrak{m}}(S') \to H^i_{\mathfrak{m}}(S'')$ sends $\alpha_i$ to 0. This implies that $H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S'')$ is the 0 map and we finish the proof. $\square$

## 3. Proof strategy

3.1. **Notations.** [[Bha21, §3.1 and Notation 4.9]] NotationXPn

In this section, we fix the following notations.

(a) Let $V$ be a $p$-torsion-free $p$-henselian excellent DVR with residue field $k$ and let $\overline{V}$ is the absolute integral closure of $V$.

(b) Fix an integer $n \geq 1$. Let $X := \mathbb{P}^n_{\overline{V}}$ be the $n$-dimensional projective space over $\overline{V}$. We use this $X$ in the rest of this section.

(c) For $X = \mathbb{P}^n_{\overline{V}}$, we take the *normalization* $\pi\colon X^+ \to X$ *of* $X$ *in* $\overline{K}$ in the sense of [GW10, Definition 12.42].

(d) As in the last paragraph of [Bha21, Notation 4.9], for quasi-compact and quasi-separated maps $f\colon Y \to X$, (for example, $f$ is a proper), we sometimes regard any sheaves on $Y$ as sheaves on $X$ via derived pushforward, that is, $\mathscr{F} \in D_{qc}(X)$ denotes $Rf_*\mathscr{F}$ for any $\mathscr{F} \in D_{qc}(Y)$ (see [Sta, 08D5]). DerivedPushforward

3.2. **Alterations over $X$.**

FiniteProper

**Definition 3.1** (Finite and proper map of schemes [Sta, 01WG] and [Sta, 01W1], or [GW10, Proposition and Definition 12.9] and [GW10, Definition 12.55])**.** Let $f\colon X' \to X$ be a map of schemes. We say that $f$ is *finite* if $f$ is affine and if, for every affine open subset $\mathrm{Spec}(B) = U \subseteq X$ with inverse image $\mathrm{Spec}(A) = f^{-1}(U) \subseteq X'$, the associated ring map $B \to A$ is finite.

We say that $f$ is *proper* if $f$ is separated, finite type, and universally closed.

GenericallyFinite

**Definition 3.2** (Generically finite map of schemes [Sta, 02NX])**.** Let $X'$ and $X$ be integral schemes and let $K(X')$ and $K(X)$ be function fields of $X'$ and $X$ each other. Let $f\colon X' \to X$ be locally of finite type map of schemes. Assume that $f$ is dominant.

We say that $f$ is *generically finite* if the following equivalent conditions are satisfied:

(1) The extension $K(X) \subseteq K(X')$ has transcendence degree 0.
(2) The extension $K(X) \subseteq K(X')$ is a finite extension.
(3) There exists a non-empty affine open subset $U' \subseteq X'$ and $U \subseteq X$ such that $f(U') \subseteq X$ and the restriction map $f|_{U'}\colon U' \to U$ is finite.
(4) The fiber $f^{-1}(\eta) \subseteq U'$ of the unique generic point $\eta$ of $X$ consists only of the unique generic point of $X'$.

If moreover $f$ is separated or if $f$ is quasi-compact, then these are also equivalent to

(1) there exists a non-empty affine open subset $U \subseteq X$ such that $f^{-1}(U) \to U$ is finite.
(2) There exists a non-empty open subset $U \subseteq X$ such that $f^{-1}(U) \to U$ is finite.

Alteration

**Definition 3.3** (Alterations [de 96, 2.20] and [Sta, 0AB0])**.** Let $X$ be an integral scheme. An *alteration* $X'$ *of* $X$ is an integral scheme $X'$ together with a map of schemes $f: X' \to X$ which is a proper dominant and generically finite.

Since every proper map is (universally) closed, alterations are actually surjective.

AlterationCategory

**Definition 3.4** (Alterations over $X = \mathbb{P}^n_{\overline{V}}$ [Bha21, Definition 4.10])**.** Fix a canonical map of schemes $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(K) \to X$. We use the following categories.

(1) Let $\mathcal{P}_X$ be the category of pairs

$$(f_Y : Y \to X, \eta_Y : \mathrm{Spec}(\overline{K}) \to Y), \tag{3.1}$$

where $Y$ is a proper integral $\overline{V}$-scheme, $f_Y$ is an alteration, and $\eta_Y$ is a map of $X$-schemes. For conveniense, we will simply write $(f_Y : Y \to X) \in \mathcal{P}_X$ or even $Y \in \mathcal{P}_X$. The map $f: Y' \to Y$ of $\mathcal{P}_X$ is the map of schemes $f: Y' \to Y$ which has natural commutativity.

(2) Let $\mathcal{P}_X^{fin}$ be the full subcategory spanned by those $Y \in \mathcal{P}_X$ such that $f_Y : Y \to X$ is finite (not only on some open subset). Since any finite map is proper and generically finite, finite alteration maps are equivalent to finite surjective maps.

(3) Let $\mathcal{P}_X^{ss}$ be the full subcategory sppaned by those $Y \in \mathcal{P}_X$ such that the $p$-adic completion $\widehat{Y}$ of $Y$, which is a $p$-adic formal $\widehat{\overline{V}} = \mathcal{O}_C$-scheme, is semistable in the sense of [ČK19, §1.5].

CofinalProperty

**Theorem 3.5** (Existence of semistable alterations by de Jong [Bha21, Theorem 4.15])**.** *The category $\mathcal{P}_X^{ss}$ is cofinal in $\mathcal{P}_X$.*

CofinalPX

**Theorem 3.6** (Cofinarity of some objects [Bha21, Theorem 4.19])**.** *We have the following compatibility of some ind-objects. (Cofinality of finite maps and vanishing of differential forms): The following natural maps of ind-objects in $D_{qc}(X_{p=0})$ are all isomorphisms and they all have colimit $\mathcal{O}_{X^+}/p$:*

$$\{\mathcal{O}_Y/p\}_{Y \in \mathcal{P}_X^{fin}} \xrightarrow{a} \{\mathcal{O}_Y/p\}_{Y \in \mathcal{P}_X} \xleftarrow{b} \{\mathcal{O}_Y/p\}_{Y \in \mathcal{P}^{ss}/p} \xrightarrow{c} \{\Delta_Y^{\log}/(p,d)\}_{Y \in \mathcal{P}_X^{ss}} \tag{3.2}$$

*where $a$ and $b$ are obtained by the tower of categories $\mathcal{P}_X^{fin} \subseteq \mathcal{P}_X \supseteq \mathcal{P}_X^{ss}$ and $c$ is the Hodge-Tate structure map $\mathcal{O}_Y/p \to \Delta_Y^{\log}/(p,d)$.*

3.3. **Key lemmas.** We use the following "equational lemma".

EquationalLem

**Lemma 3.7** ("Equational lemma" [Bha21, Lemma 4.32])**.** *Let $x \in X = \mathbb{P}^n_{\overline{V}}$ be a closed point and let $i$ be an integer. Then the $\mathcal{O}^\flat_{X^+}$-module $H^i_x(\mathcal{O}^\flat_{X^+})$ contains no nonzero Frobenius stable finitely generated $\mathcal{O}^\flat_{X^+}$-module.*

ColimPrismSS

**Lemma 3.8** ([Bha21, §4.2])**.** *For $X = \mathbb{P}^n_{\overline{V}}$, we have*

$$\operatorname*{colim}_{Y \in \mathcal{P}_X^{ss}} \Delta_Y^{\log}/p \cong \mathcal{O}^\flat_{X^+}.$$

### 3.4. $(*)_{CM}$ condition.

starCMcond

**Definition 3.9** ($(*)_{CM}$ condition [Bha21, Definition 4.1])**.** Let $(R, \mathfrak{m})$ be an excellent normal local domain. The $(*)_{CM}$ *condition* or simply $(*)_{CM}$ is the following condition about $(R, \mathfrak{m})$:

There exists a finite extension[3] $R \hookrightarrow S$ of domains such that, for any $i < \dim(R/(p))$,
$H^i_{\mathfrak{m}}(R/(p)) \to H^i_{\mathfrak{m}}(S/(p))$ is the 0 map. starCM

GeometricResult

**Theorem 3.10** (The geometric result [Bha21, Theorem 4.2])**.** *Let $V$ be a $p$-henselian $p$-torsion-free excellent DVR and let $Y$ be a flat normal $V$-scheme such that finite type over $V$. Then, for any point $y \in Y_{p=0}$, the local ring $\mathcal{O}_{Y,y}$ satisfies $(*)_{CM}$.*

*In particular, any essentially finitely generated normal local $V$-algebra $R$ satisfies $(*)_{CM}$.*

**Remark 3.11** (Proof strategy)**.** To use "equational lemma" (see Lemma 3.7), we must reduce to the case of $\mathcal{O}^{\flat}_{X^+}$, which is a colimit of prismatic complexes (see Lemma 3.8).

(1) $(M_k)$ and Lemma 2.2 shows the finiteness of some image of $\{H^i_x(\mathcal{O}_Z/p)\}$ as in Claim 3.17.
(2) By using Proposition 3.15, $H^i_x(\mathbb{\Delta}^{\log}_Y/p)$ is $d^c$-torsion up to semistable alteration, precisely Claim 3.18
(3) By using Lemma 3.16, we show the finiteness of some image of $\{H^i_x(\mathbb{\Delta}^{\log}_Y/p)[d^c]\}$, precisely Claim 3.19.
(4) Because of Lemma 3.8, $H^i_x(\mathcal{O}^{\flat}_{X^+})$ is 0 by using "equational lemma" Lemma 3.7 and the above two claim (Claim 3.18 and Claim 3.19).
(5) By Bockstein sequence we can show that $H^{i-1}_x(\mathcal{O}_{X^+}/p) \cong H^i_x(\mathcal{O}^{\flat}_{X^+}) = 0$.
(6) Again by using Claim 3.17, we can prove Theorem 3.10.

### 3.5. A Reduction step.

**Definition 3.12** ($(M_n)$ and $(P_n)$ conditions [Bha21, Definition 4.6])**.** For any integer $n \geq 1$, we define the following properties:

$(M_n)$ For any $p$-henselian $p$-torsion-free excellent DVR $V$, any flat normal finite type $V$-scheme $Y$, and any point $y \in Y_{p=0}$ with $\dim(\mathcal{O}_{Y,y}) = n+1$[4], the local ring $\mathcal{O}_{Y,y}$ satisfies $(*)_{CM}$.

$(P_n)$ For any $p$-henselian $p$-torison-free excellent DVR $V$, any closed point $x \in \mathbb{P}^n_V$, and any finite extension $\mathcal{O}_{\mathbb{P}^n_V,x} \hookrightarrow R$ of normal domains with $\dim(R/(p)) = n$, the normal domain $R$ satisfies $(*)_{CM}$, that is, there exists a finite extension $R \hookrightarrow S$ of domains such that $H^i_x(R/(p)) \to H^i_x(S/(p))$ is the 0 map for any $i < \dim(R/(p)) = n$.

[3]That is, $S$ is an integral domain and $R \hookrightarrow S$ is a finite injective map of domains.

[4]That is, the local ring $\mathcal{O}_{Y,y}$ has relative dimension $n$ over $V$

**Lemma 3.13** (Reduce to $\mathbb{P}^n_V$ [Bha21, Lemma 4.7]). *Fix an integer $n \geq 1$. Then the following are equivalent:*

*(1) $(M_k)$ holds true for all $1 \leq k \leq n$.*
*(2) $(P_k)$ holds true for all $1 \leq k \leq n$.*

*That is, to prove the following Theorem 3.10, we can reduce to the case of $X = \mathbb{P}^n_V$ (or its finite normal extension).*

Mk=Pk

**Theorem 3.14** (The geometric result: strong form [Bha21, Theorem 4.27]). *The equivalent conditions $(M_n)$ and $(P_n)$ hold true for all $n \geq 1$.*

## 3.6. Some lemmas.

FactorsCohoCM

**Proposition 3.15** ([Bha21, Proposition 4.22]). *There exists an integer $c = c(n)$ only depending on $n = \dim(X)$ such that, for any $Y \in \mathcal{P}^{ss}_X$, there is a map $f\colon Y' \to Y$ in $\mathcal{P}^{ss}_X$ and $K \in D_{comp,qc}(X, \Delta_X/p)$ such that the following holds:*

*(1) Take the pullback $f^*\colon \Delta^{\log}_Y/p \to \Delta^{\log}_{Y'}/p$. Then $d^c f^*$ factors over $K$ in $D_{comp}(X_{p=0}, A_{\inf}/p)$, that is, we have a following commutative diagram*

$$\begin{array}{ccc} \Delta^{\log}_Y/p & \xrightarrow{f^*} & \Delta^{\log}_{Y'}/p \\ {\scriptstyle\exists}\downarrow & & \downarrow{\scriptstyle \times d^c} \\ K & \xrightarrow{\exists} & \Delta^{\log}_{Y'}/p \end{array}$$

*in $D_{comp}(X_{p=0}, A_{\inf}/p)$.*

*(2) The quotient $K/d \in D_{qc}(X_{p=0})$[5] is cohomologically CM, that is, $R\Gamma_x((K/d)_x) \in D^{\geq \dim(\mathcal{O}_{X_{p=0},x})(=n-\dim(\overline{\{x\}}))}(\mathcal{O}_{X_{p=0},x})$ for any $x \in X_{p=0}$.*

PassingOPrism

**Lemma 3.16** (Passing $\mathcal{O}_Y$ to $\Delta_Y$ [Bha21, Lemma 4.25]). *Fix a closed point $x \in X$. Assume that, for any $Y \in \mathcal{P}^{ss}_X$, there exists a map $Y' \to Y$ in $\mathcal{P}^{ss}_X$ such that the map of $\mathcal{O}_{X_{p=0},x}$-modules (defined in [Sta, 0A39])*

$$H^i_x((Rf_{Y,*}\mathcal{O}_Y)/p) \longrightarrow H^i_x((Rf_{Y',*}\mathcal{O}_{Y'})/p) \tag{3.3}$$

*induced from the map of $\mathcal{O}_X$-algebras $Rf_{Y,*}\mathcal{O}_Y \to Rf_{Y',*}\mathcal{O}_{Y'}$ factors over a finitely presented $\overline{V}^\flat/(d)$-module for $i < n$.*

*Then, for any $Y \in \mathcal{P}^{ss}_X$ and any integer $c \geq 1$, there is a map $Y'' \to Y$ in $\mathcal{P}^{ss}_X$ such that the map of $\mathcal{O}_X/(p)$-modules*

(3.4)
$$H^i_x(\Delta^{\log}_Y/(p,d^c)) := R^i\Gamma^{\Delta}_{\{x\}}(\Delta^{\log}_Y/(p,d^c)) \longrightarrow R^i\Gamma^{\Delta}_{\{x\}}(\Delta^{\log}_{Y''}/(p,d^c)) =: H^i_x(\Delta^{\log}_{Y''}/(p,d^c))$$

[5]By the definition of $D_{comp,qc}(X, \Delta_X/p)$ in Section 3.1, the derived quotient $K/d$ is in the (usually defined) cohomologically quasi-coherent derived category $D_{qc}(X_{p=0})$.

*induced from the map of* $\Delta_X$*-complexes* $\Delta^{\log}_Y/p \to \Delta^{\log}_{Y''}/p$ *factors over a finitely presented* $\overline{V}^\flat$*-module for* $i < n$*.*

### 3.7. **Sketch of proof.**

*Sketch of Proof of Theorem 3.14.* We prove that $(P_k)$ holds true for all $1 \leq k \leq n$ by induction $n$. If $n = 1$, $R$ is a normal ring of dimension 2. Then $R$ is Cohen-Macaulay (for example, by using Serre's criteria), the 0-th local cohomology $H^0_x(R/(p))$ is itself 0.

Assume that $(P_k)$ (and hence $(M_k)$ by Theorem 3.14) hold true for $k < n$ and we show that $(P_n)$. Fix a closed point $x \in \mathbb{P}^n_{V/(p)}$, the special fibre of $\mathbb{P}^n_V$ over $V$. $x$ is corresponding to a closed point of $X_{p=0}$. We start the following reduction steps.

(1) (Reduction to $\overline{V}$-algebra): It suffices to show that

for any $Y \in \mathcal{P}^{ss}_X$, there exists a map $Y' \to Y$ in $\mathcal{P}^{ss}_X$ such that the induced map

$$H^i_x(\mathcal{O}_Y/p) \longrightarrow H^i_x(\mathcal{O}_{Y'}/p)$$

is the 0 map for all $i < n$.

To prove this, we need the following claims.

ImageFPO

**Claim 3.17.** *For any* $Z \in \mathcal{P}^{ss}_X$*, there exists a map* $Z' \to Z$ *in* $\mathcal{P}^{ss}_X$ *such that the induced map*

$$H^i_x(\mathcal{O}_Z/p) \longrightarrow H^i_x(\mathcal{O}_{Z'}/p)$$

*has image contained in a finitely presented* $\overline{V}^\flat$*-module for all* $i < n$*.*

*Proof.* Applying Lemma 2.2 under our assumptions $(M_k)$ for all $1 \leq k < n$. □

By using Lemma 3.16, for any $Y \in \mathcal{P}^s s_X$ and for any $c \geq 1$, there exists a map $Y' \to Y$ in $\mathcal{P}^{ss}_X$ such that the induced map

$$H^i_x(\Delta^{\log}_Y/(p, d^c)) \longrightarrow H^i_x(\Delta^{\log}_{Y'}/(p, d^c))$$

has image contained in a finitely presented $\overline{V}^\flat$-module.

ImageTorsion

**Claim 3.18** (Image is bounded $d$-torsion [Bha21, Claim 4.28])**.** *For any* $Y \in \mathcal{P}^{ss}_X$*, there exists an integer* $c = c(X) \geq 1$ *depending only on* $X$ *and a map* $Y' \to Y$ *in* $\mathcal{P}^{ss}_X$ *such that the induced map*

$$H^i_x(\Delta^{\log}_Y/p) \longrightarrow H^i_x(\Delta^{\log}_{Y'}/p)$$

*has image annihilated by* $d^c$ *for all* $i < n + 1$*.*

*Proof.* Under Claim 3.17, applying [Bha21, Lemma 4.25]. □

ImageTorsionPartFG

**Claim 3.19** ([Bha21, Claim 4.29])**.** *For any $c \geq 1$ and any $Y' \in \mathcal{P}_X^{ss}$, there exists a map $Y'' \to Y'$ in $\mathcal{P}_X^{ss}$ such that the induced map on $d^c$-torsion submodule*

$$H_x^i(\Delta_{Y'}^{\log}/p)[d^c] \longrightarrow H_x^i(\Delta_{Y''}^{\log}/p)[d^c]$$

*has image contained in a finitely generated $\overline{V}^\flat$-submodule of the tareget for all $i < n+1$.*

*Proof.* By Bockstein sequence, applying Theorem 3.6 and Claim 3.17. $\square$

In particular, there exists an integer $c \geq 1$ such that, for any $Y \in \mathcal{P}_X^{ss}$, there exist maps $Y'' \to Y' \to Y$ in $\mathcal{P}_X^{ss}$ with the following commutative diagram:

$$\begin{array}{ccccc}
H_x^i(\Delta_Y^{\log}/p) & \longrightarrow & H_x^i(\Delta_{Y'}^{\log}/p) & \longrightarrow & H_x^i(\Delta_{Y''}^{\log}/p) \\
& \searrow^{\exists} & \uparrow & & \uparrow \\
& & H_x^i(\Delta_{Y'}^{\log}/p)[d^c] & \longrightarrow & H_x^i(\Delta_{Y''}^{\log}/p)[d^c] \\
& & & \searrow^{\exists} & \uparrow \\
& & & & (\text{f.g. } \overline{V}^\flat\text{-submodule})
\end{array}$$

for all $i < n+1$.

Combining these claims, we can show that, for any $Y \in \mathcal{P}_X^{ss}$, the image of

$$H_x^i(\Delta_Y^{\log}/p) \longrightarrow H_x^i(\mathcal{O}_{X^+}^\flat)$$

is finitely generated $\overline{V}^\flat$-submodule for all $i < n+1$. By the "equational lemma" Lemma 3.7 above, this map is the 0 map. Taking the colimit over all $Y \in \mathcal{P}_X^{ss}$, then we have

$$H_x^i(\mathcal{O}_{X^+}^\flat) = 0$$

for all $i < n+1$ by Lemma 3.8. The long exact sequence of the distinguished triangle

$$\mathcal{O}_{X^+}^\flat \xrightarrow{\times p^\flat} \mathcal{O}_{X^+}^\flat \to \mathcal{O}_{X^+}^\flat/p^\flat \cong \mathcal{O}_{X^+}/p \xrightarrow{+1}$$

shows that

$$H_x^{i-1}(\mathcal{O}_{X^+}/p) \cong H_x^i(\mathcal{O}_{X^+}^\flat) = 0$$

for all $i < n+1$. Combining this vanishing with Claim 3.17 and Theorem 3.6, for any $Y \in \mathcal{P}_X^{fin}$, there exists a map $Y' \to Y$ in $\mathcal{P}_X^{fin}$ such that the induced map

$$H_x^i(\mathcal{O}_Y/p) \longrightarrow H_x^i(\mathcal{O}_{Y'}/p)$$

is the 0 map for all $i < n$. $\square$

## References


[Bha21] Bhargav Bhatt. "Cohen-Macaulayness of Absolute Integral Closures". *arXiv:2008.08070 [math]* (2021). arXiv: 2008.08070. URL: http://arxiv.org/abs/2008.08070.

[ČK19] Kęstutis Česnavičius and Teruhisa Koshikawa. "The Ainf-cohomology in the Semistable Case". *Compositio Mathematica* 155.11 (2019), pp. 2039–2128. URL: https://www.cambridge.org/core/journals/compositio-mathematica/article/atextinfcohomology-in-the-semistable-case/3004C240B5B9CC044B65BEAF42FED563#

[de 96] Aise Johan de Jong. "Smoothness, Semi-Stability and Alterations". *Publications Mathématiques de l'IHÉS* 83 (1996), pp. 51–93. URL: http://www.numdam.org/item/?id=PMIHES_1996__83__51_0.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes with Examples and Exercises.* 1st edition. Algebraic Geometry Ulrich Görtz; Torsten Wedhorn ; 1. Vieweg + Teubner, 2010.

[HH92] Melvin Hochster and Craig Huneke. "Infinite Integral Extensions and Big Cohen–Macaulay Algebras". *Annals of Mathematics* 135.1 (1992), pp. 53–89. eprint: 2946563. URL: https://www.jstor.org/stable/2946563.

[HL07] Craig Huneke and Gennady Lyubeznik. "Absolute Integral Closure in Positive Characteristic". *Advances in Mathematics* 210.2 (2007), pp. 498–504. URL: https://www.sciencedirect.com/science/article/pii/S0001870806001964.

[Quy16] Pham Hung Quy. "On the Vanishing of Local Cohomology of the Absolute Integral Closure in Positive Characteristic". *Journal of Algebra* 456 (2016), pp. 182–189. URL: https://www.sciencedirect.com/science/article/pii/S0021869316001344.

[Sta] The Stacks Project Authors. *Stacks Project.* URL: https://stacks.math.columbia.edu.



Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro, Tokyo 152-8551

*Email address*: ishizuka.r.ac@m.titech.ac.jp